Spectral Property of Dirac Operator in Chiral Quark Soliton Model

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2019. 11. 29.

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- Model and notation

Chiral Quark Soliton Model

Dirac Operator in Chiral Quark Soliton Model

- $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2$: Hilbert space of state,
- $H_0 = -i\alpha \cdot \nabla$: Free Dirac operator,

$$H_{\mathrm{CQSM}} := H_0 \otimes I_{\mathbb{C}^2} + M(\beta \otimes I_{\mathbb{C}^2}) U_F$$

- $U_F := \cos F(x) + i\gamma_5 \otimes (\boldsymbol{\sigma} \cdot \boldsymbol{n}(x)) \sin F(x)$
- $\gamma_5 := -i\alpha_1\alpha_2\alpha_3 \left(= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$
- $\boldsymbol{n} = (n_1(x), n_2(x), n_3(x)) : \mathbb{R}^3 \to \mathbb{R}^3$ s,t, $(n_1(x), n_2(x)) \neq 0$

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- Spectral Structure

SUSY QM

$\mathscr{H}$: Hilbert space, $H,\ Q$: Self-adjoint operator on $\mathscr{H}$,
Γ : Unitary operator on $\mathscr{H}$. If $\mathscr{H},\ H,\ Q$ and Γsatisfies

$$\Gamma^2 = I_{\mathscr{H}}, \quad H = Q^2, \quad \{\Gamma, Q\} = 0 \tag{1}$$

we call quadruplet$[\mathscr{H}, H, Q, \Gamma]$ as **Supersymmetric Quantum Mechanics** .

Supersymmetric aspects of CQSM

We define

$$\xi(x) = \frac{\sigma_1 n_2(x) - \sigma_2 n_1(x)}{\sqrt{n_1(x)^2 + n_2(x)^2}}, \text{ and } \Gamma(x) := \alpha_1\alpha_2\alpha_3 \otimes \xi(x), \ (x \in \mathbb{R}^3).$$

$[L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2, H^2_{\mathrm{CQSM}}, H_{\mathrm{CQSM}}, \Gamma(x)]$ is SUSY QM.

Spectral Property of CQSM

Spectral structure of CQSM

Suppose $\lim_{|x|\to\infty} F(x) = 0$, then,

$$\sigma_{\mathrm{ess}}(H_{\mathrm{CQSM}}) = (-\infty, -M] \cup [M, \infty), \quad \sigma_{\mathrm{p}}(H_{\mathrm{CQSM}}) \subset (-M, M).$$

$$V_F(x) := \sqrt{|\nabla F(x)|^2 + \sum_{k=1}^{3} |\nabla n_k(x)|^2 \sin^2 F(x)}, \quad x \in \mathbb{R}^3$$

$$C_F := \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{V_F(x)V_F(y)}{|x-y|} dxdy < \infty$$

N_H : Number of discrete eigenvalues counting multiplicities.

$$N_H \leq \frac{M^2 C_F}{2\pi^2}$$

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- Analytical Index of Fredholm Operator
- Boson part and Fermion part of Supercharge

Boson and Fermion part of $\mathscr{H}$

$[\mathscr{H}, H, Q, \Gamma]$: SUSY QM

$$\mathscr{H}_+ := \ker(\Gamma - 1), \quad \mathscr{H}_- := \ker(\Gamma + 1), \quad \mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$$

$$\mathscr{D}(Q_\pm) := \mathscr{D}(Q) \cap \mathscr{H}_\pm, \quad Q_\pm := Q\Big|_{\mathscr{D}(Q_\pm)}.$$

$$Q := \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix}, \quad \mathscr{D}(Q) = \mathscr{D}_(Q_+) \oplus \mathscr{D}(Q_-)$$

Witten Index

If Q is Fredholm operator, we can define analytical index of Q as

$$\Delta_{\mathrm{W}}(Q) := \dim\ker Q - \dim\ker Q^*.$$

Boson and Fermion part of $\mathscr{H}$

$[L^2(\mathbb{R}^3;\mathbb{C}^4), H^2_{\mathrm{CQSM}}, H_{\mathrm{CQSM}}, \Gamma]$: SUSY QM of CQSM

$$L^2(\mathbb{R}^3;\mathbb{C}^4)_+ := \ker(\Gamma - 1), \quad L^2(\mathbb{R}^3;\mathbb{C}^4)_- := \ker(\Gamma + 1),$$

$$L^2(\mathbb{R}^3;\mathbb{C}^4) = L^2(\mathbb{R}^3;\mathbb{C}^4)_+ \oplus L^2(\mathbb{R}^3;\mathbb{C}^4)_-$$

$$\mathscr{D}(H_{\mathrm{CQSM},\pm}) := \mathscr{D}(H_{\mathrm{CQSM}}) \cap \mathscr{H}_\pm,$$

$$H_{\mathrm{CQSM},\pm} := H_{\mathrm{CQSM}}\Big|_{\mathscr{D}(H_{\mathrm{CQSM},\pm})}.$$

Witten Index

From the spectral structure , H_{CQSM} is Fredholm oprator :

$$\Delta_{\mathrm{W}}(H_{\mathrm{CQSM}}) := \dim\ker H_{\mathrm{CQSM},+} - \dim\ker H^*_{\mathrm{CQSM},+}.$$

Trace Formula of Index

Let $[\mathscr{H}, H, Q, \Gamma]$ be a Supersymmetric Quantum Mechanics. We have two formula in order to calculate index :

(1) If there exists $\beta > 0$ such that $e^{-\beta H}$ is trace class, then,

$$\Delta_{\mathrm{W}}(Q) = \mathrm{Tr}(\Gamma e^{-\beta H}).$$

(2) If there exists $z \in \mathbb{C} \backslash [0, \infty)$ such that $(H - z)^{-1}$ is trace class, then,

$$\Delta_{\mathrm{W}}(Q) = -z \mathrm{Tr}(\Gamma (H - z)^{-1}).$$

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However ... , there are no $\beta > 0$ and $z \in \mathbb{C}\backslash[0, \infty)$ which satisfies assumption of formula for H_{CQSM}.

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- Polar Coordinate of Hilbert Space
- Radial Dirac Operator
- Use the assumptions in "*Multi-Soliton Solutions with Discrete Symmetries in the Chiral Quark Soliton Model*".

Let's recall Hamiltonian of This model :

$$H_{\mathrm{CQSM}} := H_0 \otimes I_{\mathbb{C}^2} + M(\beta \otimes I_{\mathbb{C}^2})U_F,$$

$$U_F = \cos F(x) + i\gamma_5 \otimes (\boldsymbol{\sigma} \cdot \boldsymbol{n}(x)) \sin F(x).$$

If F and n satisfies

$$F(x) = F(|x|), \qquad n(x) = n(|x|), \quad (x \in \mathbb{R}^3),$$

then H_{CQSM} depends only radial direction.

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then H_{CQSM} depends only radial direction.
$\rightarrow$ Use polar coordinate

Polar Coordinate of Hilbert Space and Radial Dirac Operator

$$\begin{aligned}
&L^2(\mathbb{R}^3, dx; \mathbb{C}^4) \otimes \mathbb{C}^2 \\
&\cong L^2((0,\infty), r^2 dr) \otimes \mathbb{C}^2 \otimes L^2(S^2, \sin\theta d\theta \otimes d\phi) \otimes \mathbb{C}^4
\end{aligned}$$

$$\begin{aligned}
H^r_{\mathrm{CQSM}} =& H_0 \otimes I_2 + M(\beta \otimes I_2) U_{F(r)} \\
=& \begin{pmatrix} M\cos F(r) & -\frac{\partial}{\partial r} + \frac{\kappa}{r} \\ \frac{\partial}{\partial r} + \frac{\kappa}{r} & -M\cos F(r) \end{pmatrix} \otimes I_2 \\
&+ \begin{pmatrix} M\sin F(r) & 0 \\ 0 & -M\sin F(r) \end{pmatrix} \otimes (\boldsymbol{\sigma} \cdot \boldsymbol{n})
\end{aligned}$$

Absolutely Continuous Spectrum(O.2019)

$\sigma_{\mathrm{ess}}(H^r_{\mathrm{CQSM}})$ is purely absolutely continuous.

References

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